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2008 J. Phys. A: Math. Theor. 41 392004

(http://iopscience.iop.org/1751-8121/41/39/392004)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 392004 (9pp) doi:10.1088/1751-8113/41/39/392004

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$\mathcal{PT}$-symmetric extensions of the supersymmetric Korteweg–de Vries equation

Bijan Bagchi[1] **and Andreas Fring**[2]

[1] Department of Applied Mathematics, University of Calcutta, 92, Acharya Prafulla Chandra Road, Kolkata 700 009, India
[2] Centre for Mathematical Science, City University, Northampton Square, London EC1V 0HB, UK

E-mail: BBagchi123@reddiffmail.com and A.Fring@city.ac.uk

Received 16 July 2008
Published 5 September 2008
Online at stacks.iop.org/JPhysA/41/392004

Abstract
We discuss several $\mathcal{PT}$-symmetric deformations of superderivatives. Based on these various possibilities, we propose new families of complex $\mathcal{PT}$-symmetric deformations of the supersymmetric Korteweg–de Vries equation. Some of these new models are mere fermionic extensions of the former in the sense that they are formulated in terms of superspace-valued superfields containing bosonic and fermionic fields, breaking however the supersymmetry invariance. Nonetheless, we also find extensions, which may be viewed as new supersymmetric Korteweg–de Vries equation. Moreover, we show that these deformations allow for a non-Hermitian Hamiltonian formulation.

PACS numbers: 03.65.−w, 02.30.Mv, 02.30.Ik

1. Introduction

$\mathcal{PT}$-symmetry, that is the invariance under a simultaneous parity transformation $\mathcal{P} : x \to -x$ and time reversal $\mathcal{T} : t \to -t$, is a very desirable property to have in a physical model without dissipation. For a Hamiltonian system it can be exploited to guarantee the reality of the corresponding spectrum, even though the Hamiltonian might be non-Hermitian [1–3]. However, even for non-Hamiltonian systems this principle can be used to construct interesting new complex extended models, e.g. [4–9]. See [10, 11] for a review and some recent results of this field of research.

Here we commence with an integrable model, which are well known to exhibit many extremely interesting features on the classical as well as on the quantum level. Due to their rich structure it is a very natural and common procedure to take these models as starting points and study new models closely related to them. We intend here to perturb or deform such a model in a $\mathcal{PT}$-symmetric manner. Concerning integrable models only few extensions of such type have been constructed. So far several extensions related to Calogero–Moser–Sutherland models [12–17] and the Korteweg–de Vries (KdV) equations [8, 9] have been investigated. Based on the observation that also the supersymmetric version of the KdV equation (sKdV) is $\mathcal{PT}$-symmetric, the main aim of this paper is to extend these types of analyses to this equation.

Our paper is organized as follows: in section 2 we recall some basic facts about the sKdV equation and demonstrate how the $\mathcal{PT}$-symmetry manifests itself in these equations. We exploit these observations to discuss various versions of $\mathcal{PT}$-symmetrically deformed superderivatives and demonstrate how they can be employed to construct new models. In section 3 we provide a supersymmetric Hamiltonian version of such extensions. We state our conclusions in section 4.

2. $\mathcal{PT}$-symmetric extensions of the sKdV equation

Let us first fix our notations and recall some known facts about the sKdV equation. There exist various fermionic extensions of the KdV equation in terms of superfields, which are either supersymmetric [18] or break this symmetry [19, 20] and are therefore mere fermionic extensions. We take as a starting point the former case and focus on the one-parameter family of the sKdV equation as derived first by Mathieu in [18]

$$\Phi_t = -D^6\Phi + \lambda D^2(\Phi D\Phi) + (6-2\lambda)D\Phi D^2\Phi. \tag{2.1}$$

Here λ is a real constant and $\Phi(x,\theta)$ denotes a fermionic superfield

$$\Phi(x,\theta) = \xi(x) + \theta u(x) \tag{2.2}$$

defined in terms of the fermionic (anticommuting) field $\xi(x)$, the usual bosonic (commuting) KdV field $u(x)$ and the anticommuting superspace variable θ. Furthermore D in (2.1) denotes the superderivative defined as

$$D = \theta\partial_x + \partial_\theta. \tag{2.3}$$

Expanding the superfield Φ in terms of component fields, as specified in (2.2), equation (2.1) may be re-written as a set of two coupled equations

$$u_t = -u_{xxx} + 6uu_x - \lambda\xi\xi_{xx}, \tag{2.4}$$

$$\xi_t = -\xi_{xxx} + (6-\lambda)\,\xi_x u + \lambda\xi u_x. \tag{2.5}$$

When $\lambda \to 0$ or $\xi \to 0$ equation (2.4) reduces to the standard KdV equation. In superspace the supersymmetry transformation is realized as

$$\mathcal{SUSY}: x \to x - \eta\theta, \qquad \theta \to \theta + \eta, \tag{2.6}$$

with η being an anticommuting constant. As a consequence the superfield and its components transform as

$$\mathcal{SUSY}: \Phi \to \Phi + \eta u + \theta\eta\xi_x, \qquad u \to u + \eta\xi_x, \qquad \xi \to \xi + \eta u, \tag{2.7}$$

i.e. a bosonic field is related to a fermionic one and vice versa. Equations (2.1), (2.4) and (2.5) are designed to remain invariant under the changes (2.7).

In order to see how one can deform the sKdV equation in a $\mathcal{PT}$-symmetric manner, we need to establish first how this symmetry manifests itself. We observe that equation (2.1) remains invariant under the following anti-linear symmetry transformation:

$$\mathcal{PT}: t \to -t, \qquad x \to -x, \qquad \mathrm{i} \to -\mathrm{i}, \qquad \Phi \to \mathrm{i}\Phi, \qquad D \to -\mathrm{i}D. \tag{2.8}$$

As a result of these properties of the superfield and superderivative we deduce that the component fields and the superspace variable transform as

$$\mathcal{PT} : u \to u, \qquad \xi \to \mathrm{i}\xi, \qquad \theta \to \mathrm{i}\theta. \tag{2.9}$$

These transformations leave equations (2.4) and (2.5) invariant. Note that the $\mathcal{PT}$-transformation is an automorphism and we still have $\mathcal{PT}^2 = 1$, as it should be.

Before we embark on the task of seeking $\mathcal{PT}$-symmetric extensions of equation (2.1) or its equivalent component version (2.4), (2.5), we shall define some deformations of derivatives and their supersymmetric counterparts in a more generic fashion.

2.1. Deformed (super)derivatives

In the spirit of the construction in [8, 9] we will define some new superderivatives, which respect the $\mathcal{PT}$-transformation properties (2.8). For this purpose we recall how to employ an ordinary deformed derivative $\partial_{x,\varepsilon}$ acting on some arbitrary $\mathcal{PT}$-invariant function $f(x)$

$$\partial_{x,\varepsilon} f(x) = -\mathrm{i}(\mathrm{i}f_x)^{\varepsilon}, \qquad \text{with } \varepsilon \in \mathbb{R}. \tag{2.10}$$

The case $\varepsilon = 1$ corresponds to the standard undeformed case. Note further that this deformed differential operator does not act distributively. We define higher derivatives by acting successively with ordinary derivatives on $\partial_{x,\varepsilon}$ as

$$\partial^n_{x,\varepsilon} := \partial^{n-1}_x \partial_{x,\varepsilon}. \tag{2.11}$$

Alternatively we could have introduced a nested version of (2.10) or possibly a mix of $\partial_{x,\varepsilon}$ and ∂_x in succession such as $\partial_{x,\varepsilon}(\partial_{x,\varepsilon} \cdots (\partial_{x,\varepsilon} f(x) \cdots))$ or $\partial_{x,\varepsilon}(\partial_x \cdots (\partial_{x,\varepsilon} f(x) \cdots))$. These latter possibilities do of course also not break the $\mathcal{PT}$-symmetry, but they would insinuate a much higher degree of nonlinearity than definition (2.11). More explicitly the first expressions for (2.11) read

$$\partial^2_{x,\varepsilon} f = -\mathrm{i}\varepsilon(\mathrm{i}f_x)^{\varepsilon} \frac{f_{xx}}{f_x}, \tag{2.12}$$

$$\partial^3_{x,\varepsilon} f = -\mathrm{i}\varepsilon(\mathrm{i}f_x)^{\varepsilon} \left[\frac{f_{xxx}}{f_x} + (\varepsilon - 1)\left(\frac{f_{xx}}{f_x}\right)^2 \right], \tag{2.13}$$

$$\partial^4_{x,\varepsilon} f = -\mathrm{i}\varepsilon(\mathrm{i}f_x)^{\varepsilon} \left[(2 + \varepsilon(\varepsilon - 3))\left(\frac{f_{xx}}{f_x}\right)^3 + 3(\varepsilon - 1)\left(\frac{f_{xx}}{f_x}\right)^2 f_{xxx} + \frac{f_{xxxx}}{f_x} \right]. \tag{2.14}$$

$$\vdots$$

Note that for $\varepsilon = -1/2$ the bracket in (2.13) simply becomes a Schwarzian derivative.

Obviously by construction the derivatives $\partial^n_{x,\varepsilon}$ and $\partial^n_{x,\varepsilon=1} = \partial^n_x$ transform in the same way under a $\mathcal{PT}$-transformation, i.e. $\mathcal{PT} : \partial^n_x \to (-1)^n \partial^n_x$ and $\mathcal{PT} : \partial^n_{x,\varepsilon} \to (-1)^n \partial^n_{x,\varepsilon}$, which gives rise to the simple construction principle: in a defining equation of a particular model replace ∂^n_x by $\partial^n_{x,\varepsilon}$ in order to introduce a new family of models.

Next we employ these deformations of ordinary derivatives to define a deformed version of the superderivative (2.3)

$$D_\varepsilon := \theta \partial_{x,\varepsilon} + \partial_\theta. \tag{2.15}$$

Clearly D and D_ε have the same transformation properties with regard to (2.8) and (2.9). The derivative with respect to the superspace variable is left undeformed as there is no natural deformed counterpart to this. In the deformation of the standard derivative we could show that the minus sign results from the anti-linear nature of the $\mathcal{PT}$-operator through the newly introduced factor i rather than from ∂_x. In contrast, for the derivative ∂_θ we cannot implement this feature, since in that case we have $\mathcal{PT}$: $\partial_\theta\Phi \rightarrow \partial_\theta\Phi$. Depending now on the way the higher derivatives are defined one may obtain deformations only acting on the bosonic, fermionic or possibly on both types of fields. Let us explore these possibilities.

2.1.1. $\mathcal{PT}$-symmetric superderivatives of bosonic–fermionic type. As a first option we define higher deformed superderivatives as

$$D_\varepsilon^2 := D_\varepsilon D_\varepsilon, \tag{2.16}$$
$$D_\varepsilon^n := D^{n-2} D_\varepsilon^2 \qquad \text{for } n > 2. \tag{2.17}$$

Accordingly the action on the superfield $\Phi(x,\theta)$ is computed to

$$D_\varepsilon\Phi = \theta\partial_{x,\varepsilon}\xi + u, \tag{2.18}$$
$$D_\varepsilon^2\Phi = \theta\partial_{x,\varepsilon}u + \partial_{x,\varepsilon}\xi, \tag{2.19}$$
$$D_\varepsilon^3\Phi = \theta\partial_{x,\varepsilon}^2\xi + \partial_{x,\varepsilon}u, \tag{2.20}$$
$$\vdots$$
$$D_\varepsilon^{2n-1}\Phi = \theta\partial_{x,\varepsilon}^n\xi + \partial_{x,\varepsilon}^{n-1}u, \tag{2.21}$$
$$D_\varepsilon^{2n}\Phi = \theta\partial_{x,\varepsilon}^n u + \partial_{x,\varepsilon}^n\xi. \tag{2.22}$$

This means for $n > 2$ the derivatives acting on the fermionic as well as those acting on the bosonic field are deformed. However, in general we would like to take ε to be an integer and since $\partial_{x,\varepsilon}^n\xi = -\mathrm{i}(\mathrm{i}\xi_x)^\varepsilon = 0$ for $\varepsilon = 2, 3, \ldots$ this does not appear to be an interesting choice.

2.1.2. $\mathcal{PT}$-symmetric superderivatives of fermionic type. Alternatively we may define

$$\hat{D}_\varepsilon^n := D^{n-1} D_\varepsilon, \qquad \text{for } n > 1, \tag{2.23}$$

in which case the action on the superfield $\Phi(x,\theta)$ gives

$$\hat{D}_\varepsilon\Phi = \theta\partial_{x,\varepsilon}\xi + u, \tag{2.24}$$
$$\hat{D}_\varepsilon^2\Phi = \theta u_x + \partial_{x,\varepsilon}\xi, \tag{2.25}$$
$$\hat{D}_\varepsilon^3\Phi = \theta\partial_{x,\varepsilon}^2\xi + u_x, \tag{2.26}$$
$$\vdots$$
$$\hat{D}_\varepsilon^{2n-1}\Phi = \theta\partial_{x,\varepsilon}^n\xi + \partial_x^{n-1}u, \tag{2.27}$$
$$\hat{D}_\varepsilon^{2n}\Phi = \theta\partial_x^n u + \partial_{x,\varepsilon}^n\xi. \tag{2.28}$$

Thus with this choice only the terms involving the derivatives acting on fermionic fields are $\mathcal{PT}$-symmetrically deformed, which for the reasons mentioned at the end of the last subsection is even less exciting.

2.1.3. $\mathcal{PT}$-symmetric superderivatives of bosonic type. It is clear from the above discussion that the most interesting definitions will be those just involving deformations of derivatives acting on the bosonic fields. We may achieve this by defining

$$\tilde{D}_\varepsilon^2 := D_\varepsilon D, \tag{2.29}$$

$$\tilde{D}_\varepsilon^n := D^{n-2} D_\varepsilon^2, \qquad \text{for } n > 2. \tag{2.30}$$

In this case the action on the superfield $\Phi(x, \theta)$ turns out to be

$$\tilde{D}_\varepsilon \Phi = \theta \xi_x + u, \tag{2.31}$$

$$\tilde{D}_\varepsilon^2 \Phi = \theta \partial_{x,\varepsilon} u + \xi_x, \tag{2.32}$$

$$\tilde{D}_\varepsilon^3 \Phi = \theta \xi_{xx} + \partial_{x,\varepsilon} u, \tag{2.33}$$

$$\vdots$$

$$\tilde{D}_\varepsilon^{2n-1} \Phi = \theta \partial_x^n \xi + \partial_{x,\varepsilon}^{n-1} u, \tag{2.34}$$

$$\tilde{D}_\varepsilon^{2n} \Phi = \theta \partial_{x,\varepsilon}^n u + \partial_x^n \xi. \tag{2.35}$$

Thus with this choice we have shown that only the terms involving the derivatives acting on the bosonic fields are $\mathcal{PT}$-symmetrically deformed.

According to the principle that any function which transforms as $\mathcal{PT}\colon f \to -f$ should be deformed as $f \to -\mathrm{i}(\mathrm{i}f)^\varepsilon$, we may also try to deform the superderivatives directly instead of focussing on the part of it involving the ordinary derivatives. Observing that $\mathcal{PT}\colon D\Phi \to D\Phi$, this form of deformation cannot be applied to the superderivative of first order. However, we may apply it to higher orders. We have $\mathcal{PT}\colon D^2\Phi \to -\mathrm{i}D^2\Phi$, $D^3\Phi \to -D^3\Phi$ and therefore we may consistently define

$$\check{D}_\varepsilon^n := D^n, \qquad \text{for } n = 1, 2 \tag{2.36}$$

$$\check{D}_\varepsilon^3 \Phi := -\mathrm{i}(\mathrm{i}D^3\Phi)^\varepsilon = \partial_{x,\varepsilon} u + \mathrm{i}\theta\varepsilon\partial_{x,\varepsilon-1} u \xi_{xx}, \tag{2.37}$$

$$\check{D}_\varepsilon^n := D^{n-3} \check{D}_\varepsilon^3, \qquad \text{for } n > 3. \tag{2.38}$$

Taking only $\mathcal{PT}$-symmetry as a guiding principle there are of course more possibilities. For instance, we could have also nested the derivatives as $D_\varepsilon(D_\varepsilon \cdots D_\varepsilon f) \cdots))$, or $D_\varepsilon(D_\varepsilon \cdots \check{D}_\varepsilon^3 f) \cdots))$, etc. For similar reasons as stated for the ordinary derivatives we restrain here from these choices. Alternatively we could keep the ordinary superderivatives up to some higher-order derivatives, since $D^{4n-1}\Phi \to -D^{4n-1}\Phi$ for $n \in \mathbb{N}$, but the models we are concerned with here do not involve such high order.

We may now use either of these possibilities in any of the terms in (2.1), giving rise to many different options to formulate $\mathcal{PT}$-symmetric extensions.

2.2. *Construction of new models*

We can replace the superderivatives by their deformed versions in various different terms and in addition we may introduce different deformation parameters in the higher-order derivatives. In order to explore some of these possibilities, let us first rewrite equation (2.1) as

$$\Phi_t = -D^6\Phi + 6D\Phi D^2\Phi + \lambda\Phi D^3\Phi - \lambda D\Phi D^2\Phi, \tag{2.39}$$

by using the identities $D^2(\Phi D\Phi) = D^2\Phi D\Phi + \Phi D^3\Phi$ and $D^2\Phi D\Phi = D\Phi D^2\Phi$. Note that these identities no longer hold in the deformed cases, such that we would have a different starting point when deforming (2.1) directly. As discussed above, the purely bosonic deformation is the most interesting one and we may therefore consider

$$\Phi_t = -\tilde{D}_\varepsilon^6\Phi + 6\tilde{D}_\kappa\Phi\tilde{D}_\kappa^2\Phi + \lambda\Phi\tilde{D}_\mu^3\Phi - \lambda\tilde{D}_\nu\Phi\tilde{D}_\nu^2\Phi. \tag{2.40}$$

In order to remain as generic as possible we have introduced four different deformation parameters ε, κ, μ and ν. The component version of (2.40) reads

$$u_t = -\partial^3_{x,\varepsilon} u + 6u\partial_{x,\kappa} u - \lambda\xi\xi_{xx} + \lambda u \left(\partial_{x,\mu} u - \partial_{x,\nu} u\right), \tag{2.41}$$

$$\xi_t = -\xi_{xxx} + 6u\xi_x + \lambda(\xi\partial_{x,\mu} u - u\xi_x). \tag{2.42}$$

The case $\mu = \nu$, $\kappa = 1$ constitutes a fermionic extension of the $\mathcal{PT}$-symmetric deformation of the KdV equation introduced in [9], which is obtained for $\xi \to 0$. In turn the case $\mu = \nu$, $\varepsilon = 1$ reduces for $\xi \to 0$ to the $\mathcal{PT}$-symmetric deformation of the KdV equation introduced in [8]. Noting how a deformed derivative transforms under a supersymmetry transformation

$$\mathcal{SUSY} : \partial_{x,\varepsilon} u \to \partial_{x,\varepsilon} u + \mathrm{i}\eta\varepsilon\partial_{x,\varepsilon-1} u\xi_{xx}, \tag{2.43}$$

$$\partial^3_{x,\varepsilon} u \to \partial^3_{x,\varepsilon} u + \mathrm{i}\eta\varepsilon\left(\partial^3_{x,\varepsilon-1} u\xi_{xx} + 2\partial^2_{x,\varepsilon-1} u\xi_{xxx} + \partial_{x,\varepsilon-1} u\xi_{xxxx}\right), \tag{2.44}$$

it is easily seen that equations (2.41) and (2.42) are only invariant under the supersymmetry transformations (2.7) in the case $\mu = \nu = \kappa = \varepsilon = 1$.

Instead of employing $D \to \tilde{D}_\varepsilon$ let us now use the deformation $D \to \check{D}_\varepsilon$. An interesting possibility is to deform just the first term in (2.39) and consider

$$\Phi_t = -\check{D}^6_\varepsilon \Phi + 6D\Phi D^2\Phi + \lambda\Phi D^3\Phi - \lambda D\Phi D^2\Phi. \tag{2.45}$$

Using (2.38), we find $\check{D}^6_\varepsilon \Phi = \theta\partial^3_{x,\varepsilon} u + \mathrm{i}\varepsilon\left(\partial^2_{x,\varepsilon-1} u\xi_{xx} + \partial_{x,\varepsilon-1} u\xi_{xxx}\right)$, such that the component version of (2.45) reads

$$u_t = -\partial^3_{x,\varepsilon} u + 6uu_x - \lambda\xi\xi_{xx}, \tag{2.46}$$

$$\xi_t = -\mathrm{i}\varepsilon(\partial^2_{x,\varepsilon-1} u\xi_{xx} + \partial_{x,\varepsilon-1} u\xi_{xxx}) + (6 - \lambda)u\xi_x + \lambda\xi u_x. \tag{2.47}$$

Thus equation (2.45) may also be viewed as yet another fermionic extension of the $\mathcal{PT}$-symmetric deformation of the KdV equation of [9], to which (2.46) reduces in the limits $\xi \to 0$ or $\lambda \to 0$. Interestingly this system is partially supersymmetric. We find that (2.46) remains invariant under the supersymmetry transformation (2.7), but (2.47) does not respect it.

Further interesting options are of course combinations of the above, such for instance

$$\Phi_t = -\check{D}^6_\varepsilon \Phi + 6\tilde{D}_\kappa\Phi\tilde{D}^2_\kappa\Phi + \lambda\Phi\check{D}^3_\mu\Phi - \lambda\tilde{D}_\nu\Phi\tilde{D}^2_\nu\Phi \tag{2.48}$$

or to add $\mathcal{PT}$-invariant terms which vanish in the limit $\varepsilon \to 1$. We will make use of the last possibility in order to restore full supersymmetry.

A few comments are in order: there are of course various other options, such as for instance to deform only one of the last two terms in (2.39), possibly together with the first term. This would lead to a rather strange extension, which does not reduce to any of the known $\mathcal{PT}$-extended KdV equations for $\xi \to 0$. These cases involve an additional term resulting from the fact that the original sKdV equation was constructed as a one-parameter family taking into account that the term $6uu_x$ can be supersymmetrized in various alternative ways. Further options are to use the derivatives $\hat{D}_\varepsilon$ or D_ε, which yield similar equations as above with the difference that also the derivatives acting on the ξ fields are deformed, which is, however, less interesting for the reasons mentioned above.

3. $\mathcal{PT}$ and supersymmetric non-Hermitian Hamiltonian deformations

Let us now recall the original motivation to consider $\mathcal{PT}$-symmetrically extended models, which was to exploit the feature that unbroken $\mathcal{PT}$-symmetry guarantees the reality of the corresponding spectrum. In this spirit it is highly desirable to discriminate between the models, which are Hamiltonian systems and those which are not. It is well known that the sKdV equation admits a Hamiltonian description for $\lambda = 2$, see [18], and it is interesting to investigate whether this feature survives the deformation.

Making use of the usual properties for the Berezin integral $\int \mathrm{d}\theta = 0$, $\int \mathrm{d}\theta\theta = 1$, we consider the Hamiltonian

$$H_\varepsilon = \int \mathrm{d}\mu \left[\Phi(D\Phi)^2 + \frac{1}{1+\varepsilon} D^2\Phi \check{D}_\varepsilon^3 \Phi \right] \tag{3.1}$$

$$= \int \mathrm{d}x \left[u^3 - 2\xi\xi_x u - \frac{1}{1+\varepsilon}(\mathrm{i}u_x)^{\varepsilon+1} - \frac{\varepsilon}{1+\varepsilon}(\mathrm{i}u_x)^{\varepsilon-1}\xi_x\xi_{xx} \right], \tag{3.2}$$

where we abbreviated $\int \mathrm{d}x\, \mathrm{d}\theta =: \int \mathrm{d}\mu$. This Hamiltonian is a deformed version of the sKdV Hamiltonian [18] and in addition a supersymmetrized version of the $\mathcal{PT}$-symmetrically deformed Hamiltonian [9], as it reduces to these Hamiltonians in the limits $\varepsilon \to 1$ and $\xi \to 0$, respectively. By construction H_ε is $\mathcal{PT}$-symmetric, but in addition it is also supersymmetic, which is most easily verified for the component version (3.2)

$$\mathcal{SUSY} : H_\varepsilon \to H_\varepsilon + \eta \int \mathrm{d}x \partial_x \left(\xi u^2 + \frac{\mathrm{i}^{\varepsilon-1}}{1+\varepsilon} u_x^\varepsilon \xi_x \right) = H_\varepsilon. \tag{3.3}$$

This means we can also think of H_ε as a new supersymmetric version of the KdV Hamiltonian.

Unlike as for the KdV equation, which admits a bi-Hamiltonian structure [21], see also [22], the sKdV equation is known to possess only one such structure [18], which respects supersymmetry. The Poisson brackets are defined as

$$\{F(\mu), G(\mu')\} := \int \mathrm{d}\mu_0 \frac{\delta F(\mu)}{\delta\Phi(\mu_0)} D_{\mu_0} \frac{\delta G(\mu')}{\delta\Phi(\mu_0)}. \tag{3.4}$$

Using the same Poisson bracket structure gives rise to a deformed equation of motion. With definition (3.4) we may then compute the corresponding flow as

$$\Phi_t = \{\Phi(\mu), H\} = D\frac{\delta H}{\delta\Phi} = D\left[\frac{\delta \int \mathrm{d}\mu \mathcal{H}}{\delta\Phi}\right], \tag{3.5}$$

$$= D\frac{\partial\mathcal{H}}{\partial\Phi} + D^2\frac{\partial\mathcal{H}}{\partial(D\Phi)} - D^3\frac{\partial\mathcal{H}}{\partial(D^2\Phi)} - D^4\frac{\partial\mathcal{H}}{\partial(D^3\Phi)} + \cdots. \tag{3.6}$$

For the Hamiltonian (3.1) we find

$$\Phi_t = 4D\Phi D^2\Phi + 2\Phi D^3\Phi - \frac{1}{1+\varepsilon}\left[\check{D}_\varepsilon^6\Phi + i\varepsilon D^4(D^2\Phi\check{D}_{\varepsilon-1}^3\Phi)\right], \tag{3.7}$$

with corresponding component version

$$u_t = 6uu_x - \partial_{x,\varepsilon}^3 u - 2\xi\xi_{xx} + \frac{\varepsilon-\varepsilon^2}{1+\varepsilon}\left[\partial_{x,\varepsilon-2}^3 u\xi_x\xi_{xx} + \partial_{x,\varepsilon-2}^2 u\xi_x\xi_{xxx} + \partial_x(\partial_{x,\varepsilon-2} u\xi_x\xi_{xxx})\right],$$

$$\xi_t = 4u\xi_x + 2\xi u_x - \frac{\mathrm{i}\varepsilon}{1+\varepsilon}\left(3\partial_{x,\varepsilon-1}^2 u\xi_{xx} + 2\partial_{x,\varepsilon-1} u\xi_{xxx} + \partial_{x,\varepsilon-1}^3 u\xi_x\right). \tag{3.8}$$

As we expect (3.7) and (3.8), (3.8) reduce to (2.1) and (2.4), (2.5), in the limit $\varepsilon \to 1$, respectively.

4. Conclusion

We have discussed various possibilities to introduce $\mathcal{PT}$-symmetrically deformed superderivatives. The most interesting cases are those just involving deformed derivatives acting on the bosonic field, i.e. $\tilde{D}_\varepsilon^n$ and $\check{D}_\varepsilon^n$ as defined in (2.30) and (2.38), respectively. We have demonstrated that these derivatives can be employed very systematically to construct new $\mathcal{PT}$-symmetric extensions of the sKdV equation. Most of these extensions are mere fermionic extensions, that is they involve fermionic superfields, but do not preserve the invariance under a supersymmetry transformation. Remarkably it is also possible to find genuinely supersymmetric extensions. Furthermore, these models allow for a Hamiltonian formulation. This means we may also think of these latter models as new supersymmetrized versions of the KdV equation.

Clearly with regard to these new models there are many interesting questions left to be explored. It remains to be settled whether these models possess non-trivial higher charges and if the conservation laws survive the deformation procedure [18, 23]. Possibly the new models are even integrable. Nonetheless, even when they turn out to be non-integrable one may exploit the rich properties of the underlying integrable model and treat the new models as perturbations of the former. This is somewhat similar in spirit as studying non-integrable quantum field theories as perturbations of integrable models, see e.g. [24]. Further interesting properties to investigate are the nature of the solutions these equations possess, what type of additional symmetries they allow [25], etc.

Besides these issues centered around the sKdV equation one may of course use the deformed superderivatives in other contexts to construct new $\mathcal{PT}$-symmetric deformations in the same spirit. Most immediate would be to consider the sKdV equation involving bosonic rather than fermionic superfields and its $N = 2$ version.

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